Problem Set 5 (Solutions) (Due 2/12/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work. Make use of theorems like the Triangle Inequality for Contour Integrals, the Fundamental Theorem of Contour Integrals, and the Cauchy-Goursat theorem, where applicable.

P1. Let

$$f(z) = \frac{z^2+2}{(z^2+3)(z^2+2z+1)}$$

and let C_R denote the semicircle of radius R parameterized by $z(t) = Re^{it}$ with $t \in [0, \pi]$. Show that

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0.$$

Proof. We first estimate the modulus of the integral using the triangle inequality for contour integrals. The arc length L_R of the contour C_R is $L_R = \pi R$. We need to estimate $|f(z)|$ for points$z \in C_R$. If $z \in C_R$, then $|z| = R$. Hence, by the triangle inequality for complex numbers,

$$\left|z^2+2\right| \le |z|^2 + 2 = R^2 + 2$$

Moreover, using the reverse triangle inequality for complex numbers,

$$\begin{aligned} \left|(z^2+3)(z^2+2z+1)\right| &= \left|z^2+3\right| |z+1|^2 \\ &\ge \left||z|^2 - 3\right| ||z|-1|^2 \\ &= \left|R^2-3\right| |R-1|^2 \\ &= (R^2-3)(R-1)^2. \end{aligned}$$

Since we ultimately take $R \to \infty$, the last step is justified by assuming $R > \sqrt{3}$. Together, these inequalities imply that

$$\left|\frac{z^2+2}{(z^2+3)(z^2+2z+1)}\right| \le \frac{R^2+2}{(R^2-3)(R-1)^2}.$$

By the triangle inequality for contour integrals,

$$\left|\int_{C_R} f(z)\, dz\right| \le \frac{R^2+2}{(R^2-3)(R-1)^2} \cdot L_R = \frac{\pi R(R^2+2)}{(R^2-3)(R-1)^2}.$$

The right hand side of the inequality goes to 0 as $R \to \infty$, which proves the claim. ☕

P2. Let C be a positively oriented simply closed contour and let R be the region consisting of C and its interior.

(a) Show that the area A of the region R is given by the formula[1]

$$A = \frac{1}{2i} \int_C \overline{z}\, dz.$$

(b) Compute the area A of the region enclosed by the *cardioid* C with parameterization $z(t) = \frac{1}{2} + e^{it} + \frac{1}{2}e^{2it}$ where $t \in [0, 2\pi]$.

Solution. (a) As mentioned in the lecture, we can write a contour integral in terms of the usual line integrals from vector calculus:

$$\int_C f(z)\, dz = \int_C u\, dx - v\, dy + i \int_C v\, dx + u\, dy. \qquad (*)$$

We have $f(z) = \bar{z} = x - iy$ so that $u(x, y) = x$ and $v(x, y) = -y$. Evidently, u and v have continuous first order partial derivatives everywhere on $\mathbb{C}$, and hence everywhere on the region R consisting of C and its interior. Thus, Green's theorem applies and we have

$$\begin{aligned}
\frac{1}{2i} \int_C \overline{z}\, dz &= \frac{1}{2i} \left(\int_C x\, dx + y\, dy + i \int_C (-y)\, dx + x\, dy \right) && \text{(by } (*)\text{)} \\
&= \frac{1}{2i} \iint_R 2\, dA && \text{(by Green's Theorem)} \\
&= \iint_R 1\, dA
\end{aligned}$$

From calculus, we know that the right hand side of the equation is the area of R, so the claim is proved.

(b) Following part (a), we compute

$$\begin{aligned}
2iA &= \int_C \overline{z}\, dz \\
&= \int_0^{2\pi} \left(\frac{1}{2} + e^{-it} + \frac{1}{2}e^{-2it} \right) \left(ie^{it} + ie^{2it} \right)\, dt \\
&= i \int_0^{2\pi} \frac{3}{2}e^{it} + \frac{1}{2}e^{2it} + \frac{1}{2}e^{-it} + \frac{3}{2}\, dt \\
&= 3\pi i.
\end{aligned}$$

Hence, $A = \frac{3}{2}\pi$.

P3. Let C be a closed contour and let $z_0 \in \mathbb{C}$ be a point not lying on C. The *winding number* of C about z_0 is defined by the integral

$$n(C, z_0) = \frac{1}{2\pi i} \int_C \frac{1}{z - z_0}\, dz.$$

(a) Compute $n(C_1, z_0)$ where C_1 is parameterized by $z(t) = z_0 + Re^{it},\ t \in [0, 2k\pi], k \in \mathbb{Z},\ R > 0$.

(b) Compute $n(C_2, z_0)$, where C_2 is any circle and z_0 is any point not lying on or interior to C_2.

[1] Hint: Green's theorem

(c) Let C_3 be any closed contour and z_0 any point not lying on C_3, parameterized by $z : [a,b] \to \mathbb{C}$. For any such contour, we can always find real-valued (piece-wise) differentiable functions $r, \theta : [a,b] \to \mathbb{R}$ with $r(t) > 0$ such that $z(t) = z_0 + r(t)e^{i\theta(t)}$. Compute $n(C_3, z_0)$.

(d) Give a geometric interpretation of the winding number. You must justify your interpretation.

Solution. (a) This is a straightforward computation using the definition. We have

$$\begin{aligned} n(C_1, z_0) &= \frac{1}{2\pi i}\int_0^{2k\pi} \frac{Rie^{it}}{(z_0 + Re^{it}) - z_0}\,dt \\ &= \frac{1}{2\pi}\int_0^{2k\pi} 1\,dt \\ &= k. \end{aligned}$$

So the winding number of a circle about its center is the signed number of times the circle is traversed. The sign, of course, tells you whether the circle was given the positive, or negative, orientation.

(b) In this case, the function $g(z) = \frac{1}{z-z_0}$ is analytic everywhere interior to and on C_2. By the Cauchy-Goursat theorem,

$$n(C_2, z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{1}{z-z_0}\,dz = 0.$$

So the winding number about a point not interior to the circle is 0.

(c) We use the given parameterization of C_3. Since C is a closed curve, it has the same initial and final point, that is, $z(b) = z(a)$. Since $r(t)$ is the modulus of $z(t)$, we must also have $r(b) = r(a)$.

$$\begin{aligned} n(C_3, z_0) &= \frac{1}{2\pi}\int_{C_3} \frac{1}{z-z_0}\,dz \\ &= \frac{1}{2\pi i}\int_a^b \frac{r'(t)e^{i\theta(t)} + r(t)i\theta'(t)e^{i\theta(t)}}{r(t)e^{i\theta(t)}}\,dt && \text{(product rule for } z'(t)) \\ &= \frac{1}{2\pi i}\int_a^b \frac{r'(t)}{r(t)} + \frac{1}{2\pi}\int_a^b \theta'(t)\,dt \\ &= \frac{1}{2\pi i}(\ln r(b) - \ln r(a)) + \frac{1}{2\pi}(\theta(b) - \theta(a)) \\ &= \frac{1}{2\pi}(\theta(b) - \theta(a)) && \text{(since } C \text{ is closed, } r(b) = r(a)). \end{aligned}$$

(d) Let C be a closed contour and choose a parameterization $z(t) = z_0 + r(t)e^{i\theta(t)}$. By definition, $\theta(t)$ is an argument of the complex number $r(t)e^{i\theta(t)}$. So $\theta(b) - \theta(a)$ is the change in argument of this complex number as the contour is traversed. Since C is closed, $\theta(b) - \theta(a) = 2k\pi$, where $k \in \mathbb{Z}$. Notice that $k = 0$ if and only if C encloses z_0. From part (c), we know that

$$n(C, z_0) = \frac{1}{2\pi}(\theta(b) - \theta(a)) = k.$$

So, geometrically, the winding number of a closed contour about a point is an integer representing the number of times the contour winds around that point (hence the nomenclature). The sign of k depends on the orientation of the contour.

P4. Let $f(z) = \frac{1}{z^2-1}$. Determine all possible values of the integral

$$\int_{C^{(k)}} f(z)\, dz$$

where $C^{(k)}$ is any circle, traversed k-times, and not passing through 1 and -1. You must justify your claim.

Solution. For simplicity of notation, write $V = \int_{C^{(k)}} \frac{1}{z^2-1}\, dz$. First, use the partial fraction decomposition to rewrite the integral in such a way that P3 can be applied. We have

$$\begin{aligned} V = \int_{C^{(k)}} \frac{1}{z^2-1}\, dz &= \frac{1}{2}\left(\int_{C^{(k)}} \frac{1}{z-1}\, dz - \int_{C^{(k)}} \frac{1}{z+1}\, dz\right) \\ &= \pi i \left(n(C^{(k)}, 1) - n(C^{(k)}, -1)\right). \end{aligned}$$

Now, there are four cases to consider.

(i) $C^{(k)}$ does not enclose either of the points $1, -1$. According to P3(c), $V = 0$ since both winding numbers are 0.

(ii) $C^{(k)}$ encloses 1, but not -1. Then $n(C^{(k)}, 1) = k$ and $n(C^{(k)}, -1) = 0$ by P3(c). Hence, $V = \pi i(k - 0) = k\pi i$.

(iii) $C^{(k)}$ encloses -1, but not 1. Then $n(C^{(k)}, 1) = 0$ and $n(C^{(k)}, -1) = k$ by P3(c). Hence, $V = \pi i(0 - k) = -k\pi i$.

(iv) $C^{(k)}$ encloses both 1 and -1. Then $n(C^{(k)}, 1) = k$ and $n(C^{(k)}, -1) = k$ by P3(c). Hence, $V = \pi i(k - k) = 0$.

Hence, the integral can take on the values $0, k\pi i$ and $-k\pi i$. **NOTE:** You could have calculated the integrals using the Cauchy Integral formula, instead of using winding numbers. This comment is relevant to the midterm exam. ☕

P5. Let $a, b \in \mathbb{C}$ and let C_R be the circle of radius R centered at the origin, traversed once in the positive orientation. If $|a| < R < |b|$, show that

$$\int_{C_R} \frac{1}{(z-a)(z-b)}\, dz = \frac{2\pi i}{a-b}.$$

Proof. This is (unintentionally) very similar to P4. Use the partial fraction decomposition to rewrite the integral in such a way that P3 can be applied. The condition $|a| < R < |b|$ just means that a is interior to the circle C_R, and b is exterior. Hence, $n(C_R, a) = 1$ and $n(C_R, b) = 0$. Thus, we have

$$\begin{aligned} \int_{C_R} \frac{1}{(z-a)(z-b)}\, dz &= \frac{1}{a-b}\left(\int_{C_R} \frac{1}{z-a}\, dz - \int_{C_R} \frac{1}{z-b}\, dz\right) \\ &= \frac{2\pi i}{a-b}\left(n(C_R, a) - n(C_R, b)\right) \\ &= \frac{2\pi}{a-b}. \end{aligned}$$

NOTE: You could have calculated the integrals using the Cauchy Integral formula, instead of using winding numbers. ☕

P6. Let $f(z) = \frac{1}{z^2+1}$. Determine whether f has an antiderivative on the given domain D. You must prove your claims.

(a) $D = \mathbb{C} \setminus \{i, -i\}$;

(b) $D = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$

Solution. (a) Let C be the circle of radius 1, centered at i. Notice that $-i$ is not interior to C so $n(C, -i) = 0$ while $n(C, i) = 1$. Using the partial fraction decomposition of f, and P3, we have

$$\begin{aligned} \int_C \frac{1}{z^2+1}\, dz &= \frac{i}{2}\left(\int_C \frac{1}{z+i}\, dz - \int_C \frac{1}{z-i}\, dz\right) \\ &= -\pi\,(n(C,-i) - n(C,i)) \\ &= \pi. \end{aligned}$$

But f is continuous on D so the Fundamental Theorem of Contour Integrals applies and we may conclude that f has no antiderivative on D. **NOTE:** You could have calculated the integrals using the Cauchy Integral formula, instead of using winding numbers.

(b) The only singularities of $f(z)$ are $i, -i$. Since $\operatorname{Re}(i) = \operatorname{Re}(-i) = 0$, we see that $i, -i \notin D$. Thus, $f(z)$ is analytic on D. Moreover, D is simply connected. Hence, by the Cauchy-Goursat theorem for simply connected domains, $\int_C f(z)\, dz = 0$ for any closed contour C lying in D. By the Fundamental Theorem of Contour Integrals, $f(z)$ has an antiderivative on D.